

Graphs of large linear size are antimagic

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September 15, 2014

Abstract

Given a graph $G = (V, E)$ and a colouring $f : E \mapsto \mathbb{N}$, the induced colour of a vertex v is the sum of the colours at the edges incident with v . If all the induced colours of vertices of G are distinct, the colouring is called antimagic. If G has a bijective antimagic colouring $f : E \mapsto \{1, \dots, |E|\}$, the graph G is called antimagic. A conjecture of Hartsfield and Ringel states that all connected graphs other than K_2 are antimagic. Alon, Kaplan, Lev, Roditty and Yuster proved this conjecture for graphs with minimum degree at least $c \log |V|$ for some constant c ; we improve on this result, proving the conjecture for graphs with average degree at least some constant d_0 .

1 Introduction

All graphs in this paper are simple and undirected, except where we explicitly state otherwise. By a *colouring* of a set S , we mean a function $f : S \mapsto \mathbb{N}$. For $s \in S$, $f(s)$ is called the *colour* of s . We call f a *labelling* if it is injective, and in this case $f(s)$ is called the *label* of s . For a graph G and a colouring $f : E(G) \mapsto \mathbb{N}$, the *induced colour* of a vertex v is the sum of the colours of the edges incident with v . The colouring f is called *antimagic* if the induced colours at different vertices are distinct. If a graph G admits a bijective antimagic labelling $f : E(G) \mapsto \{1, \dots, |E(G)|\}$, then we call G *antimagic*.

Hartsfield and Ringel [5] conjectured that all connected graphs on at least 3 vertices are antimagic. This problem remains open, but there are numerous partial results. Hefetz [6] proved that a graph on 3^k vertices which admits a C_3 -factor is antimagic. This was generalised by Hefetz, Saluz and Tran [7], who proved that a graph on p^k vertices admitting a C_p -factor is antimagic. Cranston [3] proved that any regular bipartite graph is antimagic. Perhaps the most significant result on antimagic graphs is that of Alon, Kaplan, Lev, Roditty and Yuster [1], who proved that there is an absolute constant c_0 such that if G is a graph on n vertices with minimum degree at least $c_0 \log n$ then G is antimagic. For more information on this and related labelling problems, see the survey paper [4].

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Our main theorem is an improvement on the result of [1]. Note that if a graph G has two isolated vertices, or any isolated edge, it cannot be antimagic. However, we shall show that if a graph G has large average degree while avoiding these trivial obstacles, G is antimagic.

Theorem 1. *There exists an absolute constant d_0 so that if G is a graph with average degree at least d_0 , and G contains no isolated edge and at most one isolated vertex, G is antimagic.*

The rest of the paper will be organised as follows. In Section 2, we prove some preliminary lemmas which will be needed during the proof of Theorem 1. Sections 3, 4 and 5 give the proof of Theorem 1. In Section 3, we shall reduce the problem of finding an antimagic labelling for a graph with large average degree to a similar problem for a graph with minimum degree at least some constant. In Section 4, we shall put a graph with large minimum degree in a special form, and in Section 5 we shall label a graph in this form. In Section 6 we shall discuss possible directions for further work.

2 Preliminary Lemmas

In this section we shall prove or recall various results which will be needed in the proof of Theorem 1. The reader who is not overly concerned with the technical details of the proof may wish only to skim this section, referring back to it as necessary during the proof.

In Subsection 2.1 we shall prove some simple results about graphs. In Subsection 2.2 we recall the definition of a total dominating set, and quote a theorem about the size of the total k -domination number of a graph with large minimum degree. These two subsections contain lemmas which will be used in Section 4, in which we take a graph with large minimum degree and partition the edges and vertices in a certain way. In Subsection 2.3 we prove four technical lemmas about edge colourings of a graph modulo k for some integer k ; these lemmas will be needed in Section 5, when we shall label the edges of a graph in the form guaranteed by Section 4.

2.1 Graph Lemmas

In this subsection, we prove two basic results on graphs. Lemma 2 is a result about colouring a graph so that every colour appears at every vertex, and Lemma 3 concerns finding a bipartition of a graph with many edges, so that each part has many edges. Corollary 4 is simply a special case of Lemma 3 — this is the form we shall find useful later.

We start with a well known lemma about equitable bipartitions of graphs. An edge-colouring of a graph G is called equitable if for every vertex v , the numbers of edges incident at v which receive each colour differ by at most 1.

Lemma 2. *Let $G = (V, E)$ be a graph with minimum degree at least $2k + 1$. Then G has an edge-colouring $f : E \mapsto \{1, 2\}$ such that every vertex is contained in at least k edges of each colour.*

Proof. We may assume G is connected; if not, we just consider each component separately. We pair up the vertices of G of odd degree, and join each pair with an extra edge to form a multigraph G' . Since all the degrees of vertices in G' are even, G' has an Eulerian circuit C — that is, a walk which begins and ends at the same vertex, and contains each edge exactly once. If any extra edge was added to G to form G' , we choose C so to start with such an edge. Now, we colour the edges of C alternately 1 and 2; each vertex is then contained in an equal number of edges of each colour, except the starting vertex of the walk, which may have 2 more edges coloured 1 than 2. When restricted to G , this colouring is equitable unless every degree is even, in which case there may be exactly one vertex with exactly 2 more edges of one colour than the other. Since G has all degrees at least $2k + 1$, in this colouring every vertex has at least k incident edges of each colour. \square

Next we shall prove a result about partitioning the vertices of a graph with many edges into two vertex classes, each having many edges — this will be used in Section 4. We define $m(n, r_1, r_2)$ to be the least r such that every graph G on n vertices with r edges has a vertex partition $V(G) = V_1 \cup V_2$ with at least r_1 edges contained in V_1 , and at least r_2 edges contained in V_2 . If even $r = \binom{n}{2}$ does not suffice, for convenience we set $m(n, r_1, r_2)$ to be $\binom{n}{2} + 1$. We bound $m(n, r_1, r_2)$ simply by considering the number of edges in each half of a random partition of V .

Lemma 3. *Let n, r_1 and r_2 be positive integers, and for $i = 1, 2$ let $p_i = \frac{\sqrt{r_i}}{\sqrt{r_1} + \sqrt{r_2}}$. Suppose that r is an integer such that*

$$rp_i^2 - \sqrt{rp_i^2 + 2rnp_i^3 - r(2n + 1)p_i^4} \geq r_i$$

holds for $i = 1, 2$. Then $m(n, r_1, r_2) \leq r$.

Proof. Let $G = (V, E)$ be a graph with n vertices and r edges — our task is to find a partition of V with at least r_1 edges in one part, and at least r_2 in the other. We take a random partition V_1, V_2 of V , with vertices placed independently with probability p_i of being in V_i . Let X_1, X_2 be the random variables corresponding to the numbers of edges contained in V_1, V_2 respectively. For $i = 1, 2$ let μ_i, σ_i and m_i be the mean, standard deviation and median of X_i respectively. It is enough to show that for $i = 1, 2$ we have $m_i > r_i$; then with positive probability we have $X_i > r_i$ for $i = 1, 2$. Now, the mean of X_i is

$\mu_i = rp_i^2$, and the variance is given by

$$\begin{aligned}
\sigma_i^2 &= \sum_{e_1, e_2 \in E(G)} \mathbb{P}(e_1 \text{ and } e_2 \in E_G(V_i)) - \mathbb{P}(e_1 \in E_G(V_i))\mathbb{P}(e_2 \in E_G(V_i)) \\
&= r(p_i^2 - p_i^4) + \sum_{v \in V} d_G(v)(d_G(v) - 1)(p_i^3 - p_i^4) \\
&< r(p_i^2 - p_i^4) + \frac{2r}{n}n(n-1)(p_i^3 - p_i^4) \\
&< r(p_i^2 - p_i^4) + 2rn(p_i^3 - p_i^4) \\
&= rp_i^2 + 2rnp_i^3 - r(2n+1)p_i^4.
\end{aligned}$$

Now, the mean and the median of a random variable differ by at most the standard deviation, and so

$$\begin{aligned}
m_i &\geq \mu_i - \sigma_i \\
&> rp_i^2 - \sqrt{rp_i^2 + 2rnp_i^3 - r(2n+1)p_i^4} \\
&\geq r_i.
\end{aligned}$$

This proves the claim. \square

We shall apply this in a specific case. If $r = an$, and $r_i = a_i n$ for $i = 1, 2$, then $p_i = \frac{\sqrt{a_i}}{\sqrt{a_1} + \sqrt{a_2}}$, and to satisfy the condition of Lemma 3 we need

$$a_i n \leq anp_i^2 - \sqrt{anp_i^2 + 2an^2p_i^3 - an(2n+1)p_i^4}.$$

Since $n > a$, it is enough that for $i = 1, 2$,

$$a_i \leq ap_i^2 - \sqrt{p_i^2 + 2ap_i^3 - 2ap_i^4}. \quad (1)$$

This holds for large enough a , proving the following corollary — this is the form of the result which we shall need in our proof of Theorem 1.

Corollary 4. *Define a function $m' : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ by letting $m'(a_1, a_2)$ be the least real a such that with $p_i = \frac{\sqrt{a_i}}{\sqrt{a_1} + \sqrt{a_2}}$ the equation (1) holds for $i \in \{1, 2\}$. Then for all positive integers n ,*

$$m(n, a_1 n, a_2 n) \leq m'(a_1, a_2)n.$$

2.2 Dominating sets

Next, we quote a bound on the total k -domination number of a graph with minimum degree at least δ . For a graph $G = (V, E)$, the *total k -domination number* of G , $\gamma_k^t(G)$, is the cardinality of the smallest vertex set $D \subseteq V$ such that $|N_G(v) \cap D| \geq k$ for each vertex $v \in V$. The following theorem was proved by Henning and Kazemi [8]:

Theorem 5. *Suppose G is a graph with minimum degree $\delta \geq k$, and $0 \leq p \leq 1$. Then*

$$\gamma_k^t(G) \leq n \left(p + \sum_{i=0}^{k-1} (k-i) \binom{\delta}{i} p^i (1-p)^{\delta-i} \right).$$

To sketch the proof of this theorem, for each vertex v we first fix a set S_v of δ neighbours of v . Then, we select a random subset R of the vertices of G by taking each with probability p . For each vertex v which has $i < k$ members of S_v in R , we add $k-i$ of its neighbours to R . The result is a k -dominating set, whose expected size is at most the bound in Theorem 5.

For positive integers k and δ , let $z(k, \delta)$ be the least real number s so that if a graph $G = (V, E)$ has minimum degree at least δ we have $\gamma_k^t(G) \leq s|V|$. For fixed k and δ large, the best bound on $\gamma_k^t(G)$ is given when $p = \frac{\ln \delta}{\delta}(1 + o(1))$, which gives a bound on $\gamma_k^t(G) \leq \frac{n \ln \delta}{\delta}(1 + o(1))$ – and so $z(k, \delta) \leq \frac{\ln \delta}{\delta}(1 + o(1))$.

2.3 Colouring graphs modulo k

In this subsection we prove four technical lemmas on colouring graphs modulo k for some integer k . These lemmas will be important in our proof of Theorem 1, where we shall often ensure that the induced sums at various vertices of a graph differ modulo k . Before we embark on the proofs of these lemmas, we introduce some terminology for colourings of graphs.

Given a graph $G = (V, E)$, an edge subset $E_1 \subseteq E$ with a colouring $f : E_1 \mapsto \mathbb{N}$, and a vertex colouring $g : V \mapsto \mathbb{N}$, we define the *partial sum* of a vertex v to be

$$s_{(G, f, g)}(v) = g(v) + \sum_{e \in E_1} f(e).$$

Now, for a vertex set $S \subseteq V$ and integers k and i , we define

$$n_{(G, f, g, S, k)}(i) = |\{v \in S : s_{(G, f, g)}(v) \equiv i \pmod{k}\}|.$$

In both these definitions, if the graph G is clear from context it will be omitted.

The next lemma is a simple result which will allow us to colour a graph G consisting of isolated edges so that the vertex sums $s_{(G, f, g)}(v)$ are not 0 or 1 modulo k , and don't take any other value modulo k too often. This result will be used to prove Lemma 7, which is an equivalent lemma for a general graph.

Lemma 6. *Let k be an odd integer with $k \geq 5$. Suppose that $G = (V, E)$ is a graph consisting only of isolated edges. Then for any colouring $g : V \mapsto \mathbb{N}$, there exists a colouring $f : E \mapsto \{0, \dots, k-1\}$ such that*

1. $n_{(G, f, g, V, k)}(0) = n_{(G, f, g, V, k)}(1) = 0$,
2. for each $2 \leq i \leq k-1$, $n_{(G, f, g, V, k)}(i) \leq |V|/(k-3) + k + 1$.

Proof. Let the edges of G be $\{e_1, \dots, e_r\}$, and write e_i as $v_{i1}v_{i2}$, such that $g(v_{i1}) - g(v_{i2}) \equiv a \pmod{k}$ for some $0 \leq a \leq (k-1)/2$. Then for $a \in \{0, \dots, (k-$

$1)/2\}$, let $G_a = (V_a, E_a)$ be the graph consisting of those edges of G for which $g(v_{i1}) - g(v_{i2}) \equiv a \pmod{k}$. We shall label each E_a separately.

Let H_a be the graph on vertex set $\{0, \dots, k-1\}$ given by joining two integers if they differ by a modulo k . In the case $a = 0$, we allow H_a to have loops. Then we choose a colouring $f_a : E_a \mapsto \{0, \dots, k-1\}$ by choosing a function $f'_a : E_a \mapsto E(H_a)$. If $e_i \in E_a$ and $f'_a(e_i) = \{u, u+a\}$, we set $f_a(e_i)$ so that $s_{(G_a, f_a, g)}(v_{i1}) \equiv u+a \pmod{k}$, and $s_{(G_a, f_a, g)}(v_{i2}) \equiv u \pmod{k}$. Then $n_{(G_a, f_a, g, V_a, k)}(i)$ is the number of edges of E_a such that $f'_a(E_a)$ contains i .

Let H'_a be the graph $H_a \setminus \{0, 1\}$. Since k is odd, the components of H_a are odd cycles (including 1-cycles if $a = 0$), and so the components of H'_a are odd length cycles, paths with an even number of vertices, and at most one path with an odd number of vertices. We pick some (not necessarily distinct) edges of H'_a in each component as follows. For an odd length cycle, we pick every edge once. For a path $v_1 \dots v_{2k}$ on an even number of vertices, we pick each of the edges $v_{2i-1}v_{2i}$ twice for $1 \leq i \leq k$. For a path $v_1 \dots v_{2k+1}$ on an odd number of vertices, we again pick each of the edges $v_{2i-1}v_{2i}$ twice for $1 \leq i \leq k$. Then we have picked at least $k-3$ edges of H'_a , such that each vertex appears in at most 2 of them. Let these edges be e'_1, \dots, e'_t . Then we define the function $f'_a : E_a \mapsto H_a$ to have its image in the set $\{e'_1, \dots, e'_t\}$, taking each element in this set at most $|E_a|/t + 1$ times. With f_a defined from f'_a as above, we have

1. $n_{(G_a, f_a, g, V_a, k)}(0) = n_{(G_a, f_a, g, V_a, k)}(1) = 0$,
2. for each $2 \leq i \leq k-1$, $n_{(G_a, f_a, g, V_a, k)}(i) \leq 2(|E_a|/t + 1) \leq |V_a|/(k-3) + 2$.

Our colouring f of $E(G)$ is defined by $f(e) = f_a(e)$ for $e \in E_a$. Then for all $0 \leq i \leq k-1$ the colouring f satisfies

$$n_{(G, f, g, V, k)}(i) = \sum_{a=0}^{(k-1)/2} n_{(G_a, f_a, g, V_a, k)}(i).$$

For $i = 0$ or 1 , this sum is zero, and for $2 \leq i \leq k-1$ the sum is at most

$$\sum_{a=0}^{(k-1)/2} |V_a|/(k-3) + 2 = |V|/(k-3) + k + 1,$$

as required. \square

Our next lemma concerns colouring the edges of a graph with no isolated vertices to achieve certain values for the $n_{(G, f, g, S, k)}(i)$ on some set S — this will be used to prove Lemma 8. The worst case is the one we have already addressed in Lemma 6, when G consists only of isolated edges and $S = V$.

Lemma 7. *Let k be an odd integer with $k \geq 5$. Suppose that $G = (V, E)$ is a graph with no isolated vertices, with $S \subseteq V$. Then for any colouring $g : V \mapsto \mathbb{N}$, there exists a colouring $f : E \mapsto \{0, \dots, k-1\}$ such that*

1. $n_{(f, g, S, k)}(0) = n_{(f, g, S, k)}(1) = 0$,

2. for each $2 \leq i \leq k-1$, $n_{(f,g,S,k)}(i) \leq |S|/(k-3) + k + 2$.

Proof. Let G_1, \dots, G_m be the components of G , ordered such that for some r we have $G_1, \dots, G_r \subseteq S$, and $G_{r+1}, \dots, G_m \not\subseteq S$. For each $1 \leq i \leq r$, e_i be any edge in $E(G_i)$. Let $V' = S \setminus \bigcup_{i=1}^r e_i$. Then we claim that for any function $t : V' \mapsto \mathbb{N}$ there is a colouring $f' : E \setminus \{e_1, \dots, e_r\} \mapsto \{0, \dots, k-1\}$ such that for each $v \in V'$ we have $s_{(f',g)}(v) \equiv t(v) \pmod{k}$.

Indeed, to construct such a colouring f' , it is enough to construct it for each component G_i . If $i > r$, let T be any spanning tree of G_i , and colour $E(G_i) \setminus T$ arbitrarily. Now, fix a vertex $v_0 \in G_i \setminus S$, and colour the edges of T by removing a leaf $v \neq v_0$ from T and colouring the corresponding edge of T , such that if $v \in S$ then the total sum $s_{(G,f',g)}(v)$ is equal to $t(v)$ modulo k . If $i \leq r$, we proceed similarly, but this time we must ensure $e_i \in E(T)$. We now colour $E(T) \setminus \{e_i\}$ by removing leaves v which are not in e_i from T , and colouring the corresponding edge of T such that $s_{(G,f',g)}(v) \equiv t(v) \pmod{k}$.

Using this, we choose $f' : E \setminus \{e_1, \dots, e_r\} \mapsto \{0, \dots, k-1\}$ so that the sums $s_{(G,f',g)}(v)$ for $v \in V'$ are not congruent to 0 or 1 modulo k , and are distributed as evenly as possible among the congruency classes in the set $\{2, \dots, k-1\}$ modulo k . In particular, for any $2 \leq i \leq k$ we have $n_{(G,f',g,V',k)} \leq |V'|/(k-2) + 1$.

We shall set f to be equal to f' on $E \setminus \{e_1, \dots, e_r\}$; it remains to colour the e_i for $1 \leq i \leq r$. We do this using Lemma 6. Let G' be the graph consisting only of the isolated edges e_i , and for $v \in V(G')$ let g' be the function $s_{(G,f',g)}(v)$. We apply Lemma 6 to the graph G' , with the vertex colouring g' . This guarantees us a colouring $f'' : E(G') \mapsto \{0, \dots, k-1\}$ of the edges e_i such that

1. $n_{(G',f'',g',V(G'),k)}(0) = n_{(G',f'',g',V(G'),k)}(1) = 0$,
2. for each $2 \leq i \leq k-1$, $n_{(G',f'',g',V(G'),k)}(i) \leq |V(G')|/(k-3) + k + 1$.

We set f to be equal to f'' on $E(G')$, and f' otherwise. For a vertex $v \in V(G')$ we have $s_{(G,f,g)}(v) = s_{(G,f',g)}(v) + s_{(G',f'',0)}(v) = s_{(G',f'',g')}(v)$ by the definition of g' . For a vertex $v \notin V(G')$, we have $s_{(G,f,g)}(v) = s_{(G,f',g)}(v)$, since f'' labels no edge which includes v . Hence for $0 \leq i \leq k-1$ we have

$$\begin{aligned} n_{(G,f,g,S,k)}(i) &= n_{(G,f,g,V(G'),k)}(i) + n_{(G,f,g,V',k)}(i) \\ &= n_{(G',f'',g',V(G'),k)}(i) + n_{(G,f',g,V',k)}(i). \end{aligned}$$

From our conditions of f'' and f' , if $i = 0$ or 1 we have $n_{(G,f,g,S,k)}(i) = 0$, and otherwise we have

$$n_{(G,f,g,S,k)}(i) \leq |V(G')|/(k-3) + k + 1 + |V'|/(k-2) + 1 \leq |S|/(k-3) + k + 2,$$

as required. \square

This allows us to prove a lemma about labelling the edges of a graph G with a vertex partition $V_1 \cup V_2$, so that for $i = 1, 2$ the sums at vertices in V_i are not equal to 0 or 1 modulo k_i , and there are not too many of these sums in any

congruency class modulo k_i . This lemma will be needed in Section 5. To prove the lemma, we shall consider a spanning subgraph H of G . Starting with a near-arbitrary labelling of $E(G)$, we shall first switch the labels on the edges in $E(H)$ with some labels on edges in V_2 , to fix sums of vertices in V_1 modulo k_1 . We shall then switch the labels on the edges in $E(H)$ with some labels on edges in V_1 , to fix sums of vertices in V_2 modulo k_2 , while not affecting our labelling modulo k_1 . For each of these steps, we shall invoke Lemma 7.

Given subsets A and $B \subseteq V(G)$, we denote by $E_G(A)$ the set of edges of G contained in A , and $E_G(A, B)$ the set of edges of G which can be written ab with $a \in A$ and $b \in B$.

Lemma 8. *Let k_1 and k_2 be coprime odd integers, both at least 5, let $G = (V, E)$ be a graph with no isolated vertices, and let L be a set of integers of size $|E|$. Suppose that there exists a partition of V into vertex classes V_1 and V_2 such that $|E_G(V_1)| \geq (k_1 k_2 + 1)|V|$, and $|E_G(V_2)| \geq (k_1 + 1)|V|$, and that L contains at least $|V| - 1$ labels in each congruency class modulo $k_1 k_2$, and at least $(k_2 + 1)(|V| - 1)$ labels in each class modulo k_1 . Then for any function $g : V \mapsto \mathbb{N}$ there exists a bijective labelling $f : E \mapsto L$ such that*

1. $n_{(G, f, g, V_1, k_1)}(0) = n_{(G, f, g, V_1, k_1)}(1) = 0$,
2. for each $2 \leq i \leq k_1 - 1$, $n_{(G, f, g, V_1, k_1)}(i) \leq |V_1|/(k_1 - 3) + k_1 + 2$,
3. $n_{(f, g, V_2, k_2)}(0) = n_{(G, f, g, V_2, k_2)}(1) = 0$,
4. for each $2 \leq i \leq k_2 - 1$, $n_{(G, f, g, V_2, k_2)}(i) \leq |V_2|/(k_2 - 3) + k_2 + 2$.

Proof. First, let H be a minimal spanning subgraph of G with no isolated vertices — so we have $|E(H)| \leq |V| - 1$. Now, let A_1 be a subset of $E_G(V_1) \setminus E(H)$, and A_2 a subset of $E_G(V_2) \setminus E(H)$, containing $k_1 k_2 |E(H)|$ and $k_1 |E(H)|$ edges respectively; these exist because $|E_G(V_1) \setminus E(H)| \geq k_1 k_2 |V| > k_1 k_2 |E(H)|$, and similarly for V_2 . We label A_1 and A_2 injectively from L such that for each $i \in \{0, \dots, k_1 k_2 - 1\}$ there are $|E(H)|$ edges in A_1 with labels congruent to i modulo $k_1 k_2$, and for each $i \in \{0, \dots, k_1 - 1\}$ there are $|E(H)|$ edges in A_2 with labels congruent to i modulo k_1 . There are enough labels of L in each congruency class to do this by our restrictions on L . Next, we assign the other labels in L injectively but otherwise arbitrarily to $E \setminus (A_1 \cup A_2)$ — let the resulting bijective labelling from E to L be f_2 .

Now, we define $g_2 : V \mapsto \mathbb{N}$ by $g_2(v) = s_{(G, f_2, g)}(v) - \sum_{v \in e \in E(H)} f_2(e)$ — that is, $s_{(G, f_2, g)}(v)$, but ignoring the labels of edges in H . Applying Lemma 7 to the graph H , with $S = V_1$, $k = k_1$, and $g = g_2$ gives us a colouring $f' : E(H) \mapsto \{0, \dots, k_1 - 1\}$ such that

1. $n_{(H, f', g_2, V_1, k_1)}(0) = n_{(H, f', g_2, V_1, k_1)}(1) = 0$,
2. for each $2 \leq i \leq k_1 - 1$, $n_{(H, f', g_2, V_1, k_1)}(i) \leq |V_1|/(k_1 - 3) + k_1 + 2$.

We use this colouring f' to define a new bijective labelling $f_1 : E \mapsto L$ as follows. For every edge $e \in E(H)$, we choose an edge $a(e) \in A_2$ such that $f_2(a(e)) \equiv$

$f'(e) \pmod{k_1}$. We choose the $a(e)$ to be distinct — this is possible, since for each $i \in \{0, \dots, k_1 - 1\}$ there are $|E(H)|$ edges $e' \in A_2$ with $f_2(e') \equiv i \pmod{k_1}$. Now, for each $e \in E(H)$, we set $f_1(e) = f_2(a(e))$, and $f_1(a(e)) = f_2(e)$, and for edges not in $E(H)$ or the image of a we set $f_2 = f_1$.

To construct this colouring from f_2 , we have taken some pairs of edges, with no edge appearing in two pairs, and swapped the labels on each pair. Hence the labels used by f_1 are exactly the same as those used by f_2 , and so f_1 is a bijective labelling from $E \mapsto L$. By our choice of g_2 , $s_{(G, f_1, g)}(v) \equiv s_{(H, f', g_2)}(v) \pmod{k_1}$ for each $v \in V_1$, and so f_1 satisfies Conditions 1 and 2 of the lemma.

We proceed similarly to change the sums at vertices of V_2 modulo k_2 — but this time we shall also ensure we do not change the labelling modulo k_1 . We define $g_1 : V \mapsto \mathbb{N}$ by $g_1(v) = s_{(G, f_1, g)}(v) - \sum_{v \in e \in E(H)} f_1(e)$. Applying Lemma 7 to the graph H , with $S = V_2$, $k = k_2$, and $g = g_1$ gives us a colouring $f'' : E(H) \mapsto \{0, \dots, k_2 - 1\}$ such that

1. $n_{(H, f'', g_1, V_2, k_2)}(0) = n_{(H, f'', g_1, V_2, k_2)}(1) = 0$,
2. for each $1 \leq i \leq k_2 - 1$, $n_{(H, f'', g_1, V_2, k_2)}(i) \leq |V_2|/(k_2 - 1) + k_2 + 2$.

We use this colouring f'' to define a new bijective labelling $f : E \mapsto L$ as follows. For every edge $e \in E(H)$, we choose an edge $a(e) \in A_1$ such that $f_1(a(e)) \equiv f''(e) \pmod{k_2}$, but now we also insist that $f_1(a(e)) \equiv f_1(e) \pmod{k_1}$. We choose the $a(e)$ to be distinct — this is possible, since for each $i \in \{0, \dots, k_1 k_2 - 1\}$ there are $|H|$ edges $e' \in A_1$ with $f_1(e') \equiv i \pmod{k_1 k_2}$. Now, for each $e \in E(H)$, we set $f(e) = f_1(a(e))$, and $f(a(e)) = f_1(e)$, and for edges not in $E(H)$ or the image of a we set $f = f_1$.

As before, to construct f from f_1 , we have taken some pairs of edges and swapped the labels on each pair, and again no edge appears in two pairs. Hence the labels used by f are exactly the same as those used by f_1 , and so f is also a bijective labelling from $E \mapsto L$. By our choice of g_1 , $s_{(G, f, g)}(v) \equiv s_{(H, f'', g_1)}(v) \pmod{k_2}$ for each $v \in V_2$, and so f satisfies Conditions 3 and 4 of the lemma. Since the labellings f_1 and f are identical viewed modulo k_1 , f also satisfies Conditions 1 and 2. \square

The final lemma of the section is another simple technical lemma, which concerns labelling a graph with a vertex partition; this lemma will be used in Section 5. The proof is similar in style to that of Lemma 7.

Lemma 9. *Let k_1 and k_2 be integers with $k_1 \geq 5$. Let $G = (V, E)$ be a graph with a vertex partition into vertex sets A and B , and let B' be a set of vertices contained in B . Suppose that every vertex in A has at least two edges to vertices in B , and that every vertex in B' has at least one edge to a vertex in A . Suppose further that L is a set of at least $|E| + k_1 k_2 (2|A| + |B'|)$ integers, containing at least $2|A| + |B'|$ representatives of each congruency class modulo $k_1 k_2$. Then for any functions $g : V \mapsto \mathbb{N}$ and $t : A \mapsto \mathbb{N}$, there is an injective labelling $f : E \mapsto L$ such that*

1. for each $v \in A$, $s_{(f, g)}(v) \equiv t(v) \pmod{k_1 k_2}$,

$$2. n_{(f,g,B',k_1)}(0) = n_{(f,g,B',k_1)}(1) = 0,$$

$$3. \text{ for each } 2 \leq i \leq k_1 - 1, n_{(f,g,B',k_1)}(i) \leq |B'|/(k_1 - 4) + 2k_1 - 3.$$

Proof. Let $E' \subseteq E$ be a set of edges which contains at least 2 edges to B from every vertex of A , and at least 1 edge to A from every vertex of B' , and has at most $2|A| + |B'|$ edges. Let G' be the graph (V, E') . Also, let $L' \subseteq L$ be a set of $k_1 k_2 (2|A| + |B'|)$ labels, with $2|A| + |B'|$ labels in each congruency class modulo $k_1 k_2$. Let $f' : E \setminus E' \mapsto L \setminus L'$ be an arbitrary injective mapping; such a mapping exists, as $|L| \geq |E| + |L'|$. For $v \in V$, let $g'(v) = s_{(G,f',g)}(v)$.

Now, L' contains $2|A| + |B'|$ labels in each congruency class modulo $k_1 k_2$, whereas E' contains at most $2|A| + |B'|$ edges, so we can label E' however we wish modulo $k_1 k_2$ while labelling injectively from L' . Hence to label E' we first define a colouring $f_k : E' \mapsto \{0, \dots, k_1 k_2 - 1\}$, and then assign labels of L' injectively to agree with f_k modulo $k_1 k_2$. Let the components of G' be G'_1, \dots, G'_m , ordered such that for some r we have $G'_1, \dots, G'_r \subseteq A \cup B'$, and $G'_{r+1}, \dots, G'_m \not\subseteq A \cup B'$. For each $1 \leq i \leq r$, select a vertex $a_i \in A \cap G'_i$, and two vertices b_{i1} and $b_{i2} \in B' \cap G'_i$ which are neighbours of A in G' . Let

$$V' = (A \cup B') \setminus \left(\bigcup_{1 \leq i \leq r} \{a_i, b_{i1}, b_{i2}\} \right).$$

Then we can colour $E' \setminus \bigcup_{1 \leq i \leq r} \{a_i b_{i1}, a_i b_{i2}\}$ such that the vertices in V' receive any specified sums modulo $k_1 k_2$. We do this similarly to in the proof of Lemma 7; we take a spanning tree T_i for each component G'_i of G' , and when $i \leq r$ ensure that the path $b_{i1} a_i b_{i2}$ is contained in T_i . We colour the edges not in some T_i arbitrarily, and then remove leaves which are not in $\{a_i, b_{i1}, b_{i2}\}$ from T_i , colouring the corresponding edges so that every vertex $v \in V'$ receives the desired sum modulo k .

Using this, we choose a colouring $f'_k : E' \setminus \bigcup_{1 \leq i \leq r} \{a_i b_{i1}, a_i b_{i2}\} \mapsto \{0, \dots, k_1 k_2 - 1\}$ such that vertices $v \in V' \cap A$ receive sum congruent to $t(v)$ modulo $k_1 k_2$, and vertices $v \in V' \cap B'$ receive sums which are not congruent to 0 or 1 modulo k_1 , and are split as evenly as possible between the congruency classes in the set $\{2, \dots, k_1 - 1\}$ modulo k_1 . For our final colouring f_k , we shall take $f_k = f'_k$ on the domain of f'_k .

At this stage, the uncoloured edges consist of r independent copies of P_3 , $b_{i1} a_i b_{i2}$, with $a_i \in A$ and $b_{i1}, b_{i2} \in B'$. However we colour the edges $b_{i1} a_i$ and $b_{i2} a_i$, we shall have

$$s_{(f_k, g')}(b_{i1}) + s_{(f_k, g')}(b_{i2}) - 2s_{(f_k, g')}(a_i) = s_{(f'_k, g')}(b_{i1}) + s_{(f'_k, g')}(b_{i2}) - 2s_{(f'_k, g')}(a_i),$$

and so the constraint that $s_{(f_k, g')}(a_i)$ is congruent to $t(a_i)$ modulo $k_1 k_2$ leads to a constraint of the form $s_{(f_k, g')}(b_{i1}) + s_{(f_k, g')}(b_{i2}) \equiv m_i \pmod{k_1 k_2}$ for some $m_i \in \{0, \dots, k_1 k_2 - 1\}$. For each $j \in \{0, \dots, k_1 - 1\}$, let $I_j \subseteq \{1, \dots, r\}$ be the set of those i with $m_i \equiv j \pmod{k_1}$, A'_j be $\{a_i : i \in I_j\}$, B'_j be $\bigcup_{i \in I_j} \{b_{i1}, b_{i2}\}$, and E'_j be $\bigcup_{i \in I_j} \{a_i b_{i1}, a_i b_{i2}\}$. We shall colour the edge sets E'_j independently of each other.

Writing $I_j = \{i_1, \dots, i_s\}$, we wish to pick the colours of the edges $a_{i_\ell} b_{i_\ell 1}$ and $a_{i_\ell} b_{i_\ell 2}$. Let $\ell \equiv c \pmod{k_1 - 4}$, where $c \in \{1, \dots, k_1 - 4\}$. Then we colour $a_{i_\ell} b_{i_\ell 1}$ and $a_{i_\ell} b_{i_\ell 2}$ so that $s_{(f_k, g')}(a_{i_\ell}) \equiv t(a_{i_\ell}) \pmod{k_1 k_2}$, and $s_{(f_k, g')}(b_{i_\ell 1}) \equiv d \pmod{k_1}$, where d is the c^{th} element of the set $\{0, \dots, k_1 - 1\} \setminus \{0, 1, j, j - 1\}$.

With this colouring, we have $s_{(f_k, g')}(b_{i_\ell 2}) \equiv j - d \pmod{k_1}$, and in particular $s_{(f_k, g')}(b_{i_\ell 2})$ is not congruent to 0 or 1 modulo k_1 . Also, for $k - 4$ consecutive members of I_j , we have $k - 4$ pairs $(b_{i_\ell 1}, b_{i_\ell 2})$. Of these, exactly one of the $b_{i_\ell 1}$ has $s_{(f_k, g')}(b_{i_\ell 1})$ congruent to each element of $\{0, \dots, k - 1\} \setminus \{0, 1, j, j - 1\}$ modulo k , and the same holds for the $b_{i_\ell 2}$. Hence the $n_{(f_k, g', B'_j, k_1)}(i)$ satisfy

1. $n_{(f_k, g', B'_j, k_1)}(i) = 0$ for all $i \in \{0, 1, j - 1, j\}$,
2. $n_{(f_k, g', B'_j, k_1)}(i) \leq |B'_j|/(k_1 - 4) + 2$ for all $1 \leq i \leq k_1$, $i \notin \{0, 1, j - 1, j\}$.

Then for any $2 \leq a \leq k_1 - 1$, $n_{(f_k, g', B'_j, k_1)}(a)$ is at most

$$|V' \cap B|/(k_1 - 2) + 1 + \sum_{\substack{0 \leq j \leq k_1 - 1, \\ j \notin \{a, a+1\}}} (|B'_j|/(k_1 - 4) + 2) \leq |B'|/(k_1 - 4) + 2k_1 - 3.$$

Taking any injective labelling $f : E \mapsto L$ which is equal to f' on $E \setminus E'$ and agrees with f_k modulo $k_1 k_2$ on E' , f satisfies the conditions of the lemma; indeed for all v in $A \cup B'$ we have $s_{(G, f, g)}(v) \equiv s_{(G, f_k, g')}(v) \pmod{k_1 k_2}$, so the properties we require for f follow from those we have proved for f_k . \square

3 Reduction to a minimum degree problem

In this section, our aim is to reduce the problem of producing an antimagic labelling for a graph with large average degree to a similar problem for a graph with large minimum degree. To do this, we must first recall the notion of the r -core of a graph. The r -core of a graph $G = (V, E)$, which we denote V_{c_r} , is the largest set of vertices such that every vertex $v \in V_{c_r}$ has $|E_G(\{v\}, V_{c_r})| \geq r$. The r -core of G can be obtained by successively removing vertices of G with degree at most $r - 1$; this shows that the subgraph of G induced by r -core of G contains all but at most $(r - 1)|V \setminus V_{c_r}|$ of the edges of G .

To label a graph $G = (V, E)$ with large average degree, we shall pick appropriate integers δ and k . Defining V_1 to be the δ -core of G , and $V_0 = V \setminus V_1$, we shall first label $E_G(V_0, V)$, so that the sums at vertices of V_0 are all divisible by k , and none are equal. We shall then label $E_G(V_1)$ so that the sums at vertices of V_1 are not divisible by k , and none are equal — this gives us our antimagic colouring. For the first stage, we need the following lemma:

Lemma 10. *Let k be an odd positive integer, let $G = (V, E)$ be a graph, and let V_0 and V_1 be vertex sets partitioning V . Then there is a colouring $f : E_G(V_0, V) \mapsto \{0, \dots, k - 1\}$ so that $s_{(f, 0)}(v)$ is divisible by k for every vertex $v \in V_0$, and each colour is used at most $|E_G(V_0, V)|/k + |V_0|$ times.*

Proof. It is enough to prove the lemma for a connected graph; indeed, for a general graph we can simply apply the lemma to each connected component. We split the proof into three cases. Firstly, if V_1 is non-empty, let F be a forest with edges in $E_G(V_0, V)$ which spans V_0 , and has exactly one vertex of V_1 in each component. We colour $E_G(V_0, V) \setminus E(F)$ as evenly as possible with $\{0, \dots, k-1\}$, and otherwise arbitrarily. Then there is some colouring of F such that the overall sum at every vertex in V_0 is divisible by k ; we can obtain such a colouring by succesively removing leaves v of F which are in V_0 , and colouring the corresponding edge to ensure the sum at v is divisible by k .

Secondly, if V_1 is empty and G is not bipartite, let G' be any connected subgraph of G which spans V and has exactly one cycle C , which is of odd length — so G' has $|V|$ edges. We colour $E \setminus E(G')$ as evenly as possible with $\{0, \dots, k-1\}$, and otherwise arbitrarily. Then we claim that there is some colouring of $E(G')$ so that the overall sum at every vertex in V is divisible by k . We obtain this colouring by succesively removing degree 1 vertices v from G' , colouring the corresponding edges to ensure the overall sum at v is divisible by k . We do this until we are left with only the odd cycle C ; we now need to colour the edges of C so that every vertex on it has sum divisible by k . If the cycle is of length r , this is equivalent to solving a system of equations

$$\begin{aligned} a_1 + a_2 &\equiv b_1 \pmod{k} \\ a_2 + a_3 &\equiv b_2 \pmod{k} \\ &\dots \\ a_r + a_1 &\equiv b_r \pmod{k}. \end{aligned}$$

Here, the a_i correspond to the colours being given to the edges of the cycle C , and the b_i to the remaining sum needed at the vertices of C to bring the sum to 0 modulo k . Since r and k are both odd, this system of equations does indeed have a solution.

Finally, suppose V_1 is empty and G is bipartite, with vertex classes A and B . Let T be a spanning tree of G , and v_0 any vertex of G — say $v_0 \in A$. We colour $E \setminus E(T)$ as evenly as possible with $\{0, \dots, k-1\}$, and otherwise arbitrarily. Then there is some colouring of T such that every vertex in V other than v_0 has induced sum divisible by k ; as ever, we obtain such a colouring by succesively removing leaves of T and colouring the corresponding edge. Let f be the colouring $f : E \mapsto \{0, \dots, k-1\}$ this gives. Since $\sum_{v \in A} s_{(f,0)}(v) = \sum_{v \in B} s_{(f,0)}(v) = \sum_{e \in E} f(e)$, the induced sum at v_0 is also divisible by k .

In each case, we have coloured all but at most $|V_0|$ of the edges of $E_G(V_0, V)$ as evenly as possible with $\{0, \dots, k-1\}$, and then coloured the remainder in some specified way. Hence each of the colours $\{0, \dots, k-1\}$ is used on at most $|E_G(V_0, V)|/k + |V_0|$ edges. \square

Now we are in a position to prove a lemma which allows us to label $E_G(V_0, V)$ so that the sums at vertices in V_0 are distinct, and all are divisible by some integer k .

Lemma 11. *Let δ and k be odd positive integers, and let $G = (V, E)$ be a graph with no isolated edges and at most one isolated vertex, with δ -core V_1 and $V_0 = V \setminus V_1$. Let L be an interval of \mathbb{N} of length at least $(\delta - 1 + 3k)|V_0|$. Then there is an injective labelling $f : E(V_0, V) \mapsto L$ such that the sum $s_{(f,0)}(v)$ is divisible by k for each $v \in V_0$, and $s_{(f,0)}(v_1) \neq s_{(f,0)}(v_2)$ for distinct vertices v_1 and v_2 in V_0 .*

Proof. By Lemma 10 we can choose a colouring $f_k : E_G(V_0, V) \mapsto \{0, \dots, k-1\}$ such that every vertex $v \in V_0$ has $s_{(f_k,0)}(v)$ divisible by k , and each colour is used at most $|E_G(V_0, V)|/k + |V_0|$ times. Now, we label $E_G(V_0, V)$ with labels from L , stepping through the edges in any order. Let $E_G(V_0, V) = \{e_1, \dots, e_r\}$. For $0 \leq i \leq r$ we define a labelling $f^i : \{e_1, \dots, e_i\} \mapsto L$, by setting $f^i = f^{i-1}$ on $\{e_1, \dots, e_{i-1}\}$, and setting $f^i(e_i) = l$ for some label l that obeys the following conditions:

1. l is in L , and $l \equiv f_k(e_i) \pmod{k}$,
2. l is not in the image of f^{i-1} ,
3. if $v \in V_0$ and $v \in e_i$, $s_{(f^{i-1},0)}(v) + l \neq s_{(f^{i-1},0)}(v')$ for any $v' \in V_0$ with $v' \notin e_i$.

We claim that there is always a label which obeys these restrictions. Indeed, for $0 \leq i \leq k-1$ let L_i be the set of labels in L which are congruent to i modulo k . We wish to label e_i with a label in $L_{f_k(e_i)}$. Since f_k uses each colour at most $|E_G(V_0, V)|/k + |V_0|$ times, the second condition rules out at most $|E_G(V_0, V)|/k + |V_0| - 1$ labels in $L_{f_k(e_i)}$. The third condition applies to at most 2 distinct vertices v , and for each rules out at most $|V_0| - 2$ labels. Hence the total number of labels in $L_{f_k(e_i)}$ which violate one of these two conditions is at most

$$|E_G(V_0, V)|/k + |V_0| - 1 + 2(|V_0| - 2) < |E_G(V_0, V)|/k + 3|V_0| - 1.$$

On the other hand, since V_0 is the complement of the δ -core of G , $|E_G(V_0, V)| \leq (\delta - 1)|V_0|$. Hence $|L| \geq |E_G(V_0, V)| + 3k|V_0|$, and so $|L_{f_k(e_i)}| \geq |E_G(V_0, V)|/k + 3|V_0| - 1$. So there is some label l which we can use at e_i .

Let the labelling this process gives be $f = f^r$. Since f agrees with f_k modulo k , every sum $s_{(f,0)}(v)$ is divisible by k for $v \in V_0$. For $v_1 \neq v_2$ vertices of V_0 , let e_j be the last edge incident with exactly one of v_1 and v_2 to be labelled; such an edge exists since G has at most one isolated vertex and no isolated edges. When e_j is labelled, Condition 3 on $f^j(e_j)$ guarantees $s_{(f^j,0)}(v_1) \neq s_{(f^j,0)}(v_2)$, and hence $s_{(f,0)}(v_1) \neq s_{(f,0)}(v_2)$. \square

This enables us to give a lemma which is sufficient for graphs with large average degree to be antimagic; Sections 4 and 5 will be devoted to the proof of this lemma. Given a graph $G = (V, E)$ and a function $g : V \mapsto \mathbb{N}$, we call a colouring $f : E \mapsto \mathbb{N}$ *g -antimagic* if $s_{(f,g)}(v_1) \neq s_{(f,g)}(v_2)$ for distinct vertices v_1 and v_2 in V .

Lemma 12. *Let k_1 and k_2 be sufficiently large odd coprime integers. Then there are constants $c = c(k_1, k_2)$ and $\delta = \delta(k_1, k_2)$ such that if $G = (V, E)$ is a graph with minimum degree at least δ , L is a set of integers of size $|E|$ containing $\{1, \dots, c|V|\}$, and g is a function $g : V \mapsto \mathbb{N}$, then there exists a g -antimagic bijective labelling $f : E \mapsto L$ such that no vertex in V has induced sum $s_{(f,g)}(v)$ divisible by $k_1 k_2$.*

In fact, the truth of this lemma for a single pair of integers k_1 and k_2 is sufficient to prove Theorem 1. For integers a and b , we define $[a, b]$ to be the set $\{n \in \mathbb{N} : a \leq n \leq b\}$.

3.1 Proof of Theorem 1 from Lemma 12

Suppose that Lemma 12 holds for some k_1 and k_2 , with constants $c = c(k_1, k_2)$ and $\delta = \delta(k_1, k_2)$. Then we claim that Theorem 1 holds for

$$d_0 = 2 \max(c, \delta - 1 + 3k_1 k_2). \quad (2)$$

Indeed, given a graph G with average degree at least d_0 , let V_1 be the δ -core of G and $V_0 = V \setminus V_1$. Now, apply Lemma 11 to the graph G , with $k = k_1 k_2$ and the label set $L' = [|E| - (\delta - 1 + 3k_1 k_2)|V_0| + 1, |E|]$. This gives us an injective labelling $f_1 : E(V_0, V) \mapsto L'$, so that $s_{(G, f_1, 0)}(v) \equiv 0 \pmod{k_1 k_2}$ for each $v \in V_0$, and $s_{(G, f_1, 0)}(v_1) \neq s_{(G, f_1, 0)}(v_2)$ for v_1 and v_2 distinct vertices in V_0 . We define $L = [1, |E|] \setminus f_1(E(V_0, V))$; so certainly $[1, nd_0/2 - (\delta - 1 + 3k_1 k_2)|V_0|] \subseteq L$. Also, note that

$$\begin{aligned} nd_0/2 &\geq n \max(\delta - 1 + 3k_1 k_2, c) \\ &\geq (\delta - 1 + 3k_1 k_2)|V_0| + c|V_1|. \end{aligned}$$

Hence L contains $[1, c|V_1|]$, and we can apply Lemma 12 to the integers k_1 and k_2 , the graph $G' = (V_1, E_G(V_1))$ and the label set L . The function g we use is $g(v) = s_{(G, f_1, 0)}(v)$ for $v \in V_1$. So from the conclusion of Lemma 12, there exists a g -antimagic bijective labelling $f_2 : E_G(V_1) \mapsto L$, so that no vertex in V_1 has $s_{(G', f_2, g)}(v)$ divisible by $k_1 k_2$. We define the labelling $f : E \mapsto [1, |E|]$ to be equal to f_1 on $E_G(V_0, V)$ and equal to f_2 on $E_G(V_1)$. Note that f is a bijective labelling from E to $[1, |E|]$; indeed, f_1 is bijective from $E(V_0, V) \mapsto f_1(E(V_0, V))$, and f_2 is bijective from $E_G(V_1) \mapsto L = [1, |E|] \setminus f_1(E(V_0, V))$.

Now, for $v \in V_1$ we have

$$s_{(G, f, 0)}(v) = s_{(G', f_2, 0)}(v) + s_{(G, f_1, 0)}(v) = s_{(G', f_2, g)}(v),$$

so the sums $s_{(G, f, 0)}(v)$ for $v \in V_1$ are distinct and not divisible by $k_1 k_2$. Also, $v \in V_0$ we have $s_{(G, f, 0)}(v) = s_{(G, f_1, 0)}(v)$, as f_2 labels no edge incident with v , so the sums $s_{(G, f, 0)}(v)$ for $v \in V_0$ are distinct and divisible by $k_1 k_2$. Hence $s_{(G, f, 0)}(v_1) \neq s_{(G, f, 0)}(v_2)$ for v_1 and v_2 distinct vertices in V , and so G is antimagic.

4 A partition of a graph with large minimum degree

Now we shall begin the proof of Lemma 12. Given a graph $G = (V, E)$ with large minimum degree, Lemma 13 shows that we can pick some vertex disjoint stars in G with a large edge set, such that removing these stars leaves many edges in G . We use this to prove Lemma 14, which guarantees we can partition V and E into a certain form. In Section 5, we shall show that a graph in this form satisfies the conclusions of Lemma 12.

We define a *star* to be a graph $S = (V, E)$ on at least 2 vertices with a distinguished vertex c such that $E = \{cv : v \in V \setminus \{c\}\}$. The vertex c is called the *centre* of S . A *star forest* is just a collection of vertex-disjoint stars. Also, recall that in Subsection 2.2, $z(k, \delta)$ was defined as the smallest real number s such that any graph with minimum degree δ and n vertices has a set of at most sn vertices with at least k edges to every vertex of G .

Lemma 13. *Let δ , n and r be positive integers with $r \leq n\delta/2$ and $\delta \geq 5$. Let $G = (V, E)$ be a graph with minimum degree at least δ and $|V| = n$. Then there exists a star forest $F_S \subseteq G$ such that the following hold:*

1. $|E_G(V \setminus V(F_S))| \geq r$,
2. $|E(F_S)| \geq n(1/2 - z(5, \delta) - 2/\delta) - 1 - r/\delta$,
3. *There is a set V_1 consisting of some of the centres of the stars in F_S , such that:*
 - *Every vertex in V_1 has at least 5 edges to $V \setminus V(F_S)$.*
 - *Every vertex in G has at least 5 edges to $(V \setminus V(F_S)) \cup V_1$.*

Proof. Note that if the lemma holds for a graph G , and G is a subgraph of a graph G' on the same vertex set, the lemma also holds for G' ; indeed, any choice of F_S and V_1 which verifies the lemma for G also do so for G' . So it is sufficient to prove the lemma for graphs G with minimum degree δ and no edges v_1v_2 for which v_1 and v_2 have degree greater than δ , since every graph with minimum degree at least δ has a subgraph satisfying this condition; indeed, such a subgraph can be obtained by successively removing edges between two vertices of degree greater than δ .

Given a graph $G = (V, E)$ of this form, let V_s be the set of vertices of degree δ , and let $V_b = V \setminus V_s$. Note that $E_G(V_b) = \emptyset$. Now, let D be a smallest set of vertices in G with $|N_G(v) \cap D| \geq 5$ for each $v \in V$; since $\delta \geq 5$, D certainly exists. By the definition of $z(k, l)$, $|D| \leq z(5, \delta)n$. Let $D_s = D \cap V_s$. We now give an algorithm for choosing our star forest F_S , as follows:

1. $V(0) = V \setminus D_s$.
2. Given $V(t)$, let $G(t)$ be the graph $(V(t), E_G(V(t)))$, the graph induced by G on vertex set $V(t)$. Let $V_s(t) = V(t) \cap V_s$, and $V_b(t) = V(t) \cap V_b$.

3. If $|E_G(V(t))| < r + n + \delta$, we terminate the algorithm, setting F_S to be the stars $\{S_1, \dots, S_t\}$ picked so far.
4. Otherwise, we pick a star $S_{t+1} \subseteq G(t)$ with centre c_{t+1} to add to our star forest. To pick the centre c_{t+1} of the star, if there is an edge $v_1 v_2 \in E_G(V(t))$ with $v_1 \in V_s(t)$ and $v_2 \in V_b(t)$, we choose any such edge and let $c_{t+1} = v_2$. Otherwise, since $E_G(V_b) = \emptyset$, $E(G(t))$ must contain an edge $v_1 v_2$ with both v_1 and v_2 in $V_s(t)$; then we choose any such edge and let $c_{t+1} = v_1$.
5. To pick the vertex set A_{t+1} for S_{t+1} , let $N = N_{G(t)}(c_{t+1})$ be the neighbourhood of c_{t+1} in $G(t)$, and write $N = \{n_1, \dots, n_l\}$. Letting l' be maximal such that $|E_G(V(t) \setminus \{c_{t+1}, n_1, \dots, n_{l'}\})| \geq r$, we set $A_{t+1} = \{c_{t+1}\} \cup \{n_1, \dots, n_{l'}\}$.
6. We set $V(t+1) = V(t) \setminus A_{t+1}$, and go to Step 2.

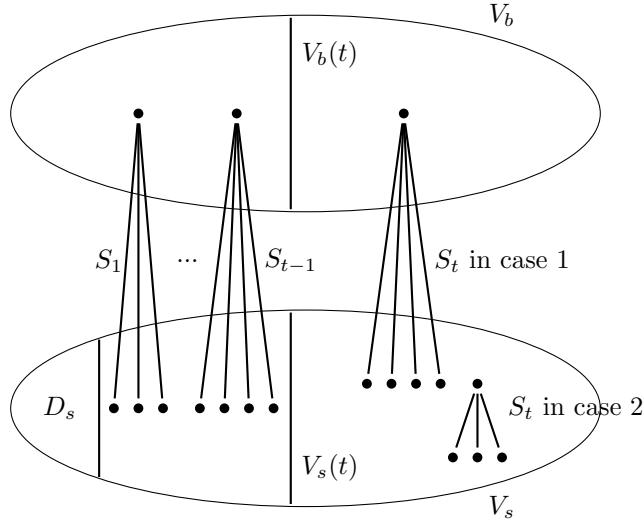


Figure 1: Picking the star S_t

This algorithm is illustrated by Figure 1. Now, let F_S be the star forest defined by this algorithm, and let $V_1 = V(F_S) \cap V_b$. We claim that F_S and V_1 satisfy the conclusions of the lemma. First, in Step 5 of the algorithm note that since $E_G(V_b) = \emptyset$ we have $N \subseteq V_s$. So the only vertices of F_S which can lie in V_b are the centres of stars, and so we do indeed have V_1 being a subset of the centres of stars in F_S .

Also, since in Step 5 of the algorithm we have $N \subseteq V_s$, all the vertices of N have degree at most δ in $G(t)$. Thus for $1 \leq i \leq l$ we have

$$|E_G(V(t) \setminus \{c, n_1, \dots, n_i\})| \geq |E_G(V(t))| - n - i\delta,$$

and in particular $|E_G(V(t)) \setminus \{c, n_1\}| \geq r$, so $l \geq 1$ and S_{t+1} has at least two vertices and is indeed a star. Also, if $V(S_{t+1}) \neq \{c_{t+1}\} \cup N$, let $V(S_{t+1}) = \{c_{t+1}, v_1, \dots, v_{l'}\}$. Then the degree of $v_{l'+1}$ in $G(t)$ is at most δ , and removing it from $G(t+1)$ reduces the number of edges in $G(t+1)$ to less than r , so we have $|E_G(V(t+1))| < r + \delta < r + \delta + n$. Hence we terminate the algorithm on the next run through, and S_{t+1} is the last star chosen.

Now we check the first two conditions of the lemma. In choosing the star S_{t+1} from $G(t)$, we ensure that $G(t+1)$ has at least r edges, so Condition 1 is satisfied. To show Condition 2 is satisfied, let the set of stars produced by the algorithm be $\{S_1, \dots, S_k\}$. Since all the vertices of $N_{G(t-1)}(c_t)$ have degree at most δ in G and hence in $G(t-1)$, for $1 \leq t \leq k$ we have

$$\begin{aligned} |E_G(V(t-1))| - |E_G(V(t))| &\leq |N_{G(t-1)}(c_t)| + (\delta - 1)(|V(S_t)| - 1) \\ &< n + \delta |E(S_t)|. \end{aligned}$$

If $t \neq k$, we also have $V(S_t) = \{c_t\} \cup N_{G(t-1)}(c_t)$, and so instead we get

$$|E_G(V(t-1))| - |E_G(V(t))| \leq \delta |E(S_t)|.$$

Hence we have

$$\begin{aligned} r + n + \delta &\geq |E_G(V(k))| \\ &= |E_G(V(0))| - \sum_{i=1}^k (|E_G(V(i-1))| - |E_G(V(i))|) \\ &\geq |E| - \delta |D_s| - \sum_{i=1}^k \delta |E(S_i)| - n \\ &\geq n\delta/2 - \delta n z(5, \delta) - \delta |E(F_S)| - n. \end{aligned}$$

Rearranging this, we obtain

$$|E(F_S)| \geq n(1/2 - z(5, \delta) - 2/\delta) - 1 - r/\delta.$$

This is the statement of the Condition 2 of the lemma. For the final condition, since $D \subseteq D_s \cup V_b \subseteq (V \setminus V(F_S)) \cup V_1$, every vertex of G has at least 5 edges to $(V \setminus V(F_S)) \cup V_1$, as required. In the case of a vertex in V_1 , these must all be to $V \setminus V(F_S)$ – indeed, $V_1 \subseteq V_b$, and so V_1 has no internal edges. \square

In the next lemma, we use the structure given by Lemma 13 to find a more precise structure in a graph G of high minimal degree. Recall that in Subsection 2.1 $m(n, r_1, r_2)$ was defined as the least integer r such that if a graph $G = (V, E)$ on n vertices has at least r edges, V can be partitioned into subsets V_1 and V_2 such that $|E_G(V_1)| \geq r_1$ and $|E_G(V_2)| \geq r_2$. Note that if $n' \leq n$ we have $m(n', r_1, r_2) \leq m(n, r_1, r_2)$; indeed, if there exists a graph on n' vertices with r edges which shows that $m(n', r_1, r_2) > r$, the same graph together with $n - n'$ isolated vertices shows that $m(n, r_1, r_2) > r$.

Lemma 14. *Let δ, n, r, r_1 and r_2 be positive integers such that $m(n, r_1, r_2) + n \leq r \leq \delta n/2$, and $\delta \geq 5$. Let $G = (V, E)$ be a graph on n vertices with minimum degree at least δ . Then there exists a star forest F_S , a vertex set V_1 consisting of some of the centres of stars in F_S , and a forest F such that:*

1. $F_S \subseteq F \subseteq G$, and F is a spanning forest for G .
2. $|E_G(V \setminus V(F_S))| \geq r$,
3. $|E(F_S)| \geq n(1/2 - z(5, \delta) - 2/\delta) - 1 - r/\delta$,
4. every component of F which is not contained in $V(F_S)$ has size at least 3,
5. for all vertices $v \in V(F_S) \setminus V_1$, $E_{F_S}(\{v\}) = E_F(\{v\})$,
6. for all vertices $v \in V \setminus V_1$, there are at least 2 edges from v to $(V \setminus V(F_S)) \cup V_1$ which are not in F ,
7. for all vertices $v \in V_1$, there are at least 2 edges from v to $V \setminus V(F_S)$ which are not in F ,
8. $V \setminus V(F_S)$ has a partition into sets U_1 and U_2 such the $E_G(U_1) \setminus E(F) \geq r_1$, and $E_G(U_2) \setminus E(F) \geq r_2$.

Proof. First, applying Lemma 13 to the graph G gives us a star forest F_S and a set V_1 consisting of some of the centres of F_S such that the conclusion of Lemma 13 applies. Next, we shall select the forest F . We define vertex sets V_0 and V_2 by $V_0 = V \setminus V(F_S)$ and $V_2 = V(F_S) \setminus V_1$; so (V_0, V_1, V_2) is a partition of V .

Let G_1 be the graph on vertex set $V_0 \cup V_1$, and edge set $E_G(V_0) \cup E_G(V_0, V_1)$. Note that by Condition 3 of Lemma 13, G_1 has minimum degree at least 5. So applying Lemma 2 to the graph G_1 gives us a colouring $f : E_G(V_0) \cup E_G(V_0, V_1) \mapsto \{1, 2\}$ with each colour appearing twice at every vertex. Now, let G_2 be the graph on vertex set $V_0 \cup V_1$ with edge set $\{e \in E_G(V_0) \cup E_G(V_0, V_1) : f(e) = 2\}$. We define F' to be any spanning forest for the graph G_2 such that the components of F' are the same as the components of G_2 , and F to be the forest with vertex set V and edge set $E(F') \cup E(F_S)$.

We claim that all the conditions of the lemma except the last now hold. For Condition 1, we need to check that F is a spanning forest of G . F is the union of a spanning forest for $V_0 \cup V_1$, together with a set of stars which span V_2 and each have at most one vertex in V_1 . Hence F is a spanning forest for V .

Conditions 2 and 3 follow immediately from the corresponding conditions in Lemma 13. For Condition 4, all vertices in V_0 have at least two edges e in $E_G(V_0) \cup E_G(V_0, V_1)$ with $f(e) = 2$, and so are in components of G_2 , and hence of F , of size at least 3. Condition 5 follows since $E(F) = E(F') \cup E(F_S)$, and F' does not have any edges incident with V_2 .

For vertices in V_2 , Condition 6 is then immediate from Condition 3 in Lemma 13; in fact, all vertices $v \in V_2$ have at least five edges to $V_0 \cup V_1$, and at most one of these edges is in F . Condition 6 also holds for vertices in V_0 , since every vertex $v \in V_0$ has 2 edges e to $V_0 \cup V_1$ with $f(e) = 1$, and these edges cannot

be in F . Similarly, Condition 7 holds because every vertex $v \in V_1$ has 2 edges e to V_0 with $f(e) = 1$.

It remains only to choose the partition (U_1, U_2) of V_0 so as to satisfy Condition 8. Let G' be the graph on vertex set V_0 , with edge set $E_G(V_0) \setminus F$. Now, $E_G(V_0)$ contains at least $r \geq m(n, r_1, r_2) + n$ edges, and F has fewer than n edges overall, so $|E(G')| = |E_G(V_0) \setminus F| \geq m(n, r_1, r_2)$. Since G' has at most n vertices, this guarantees a partition of $V(G')$ into sets U_1 and U_2 so that $|E_{G'}(U_1)| \geq r_1$ and $E_{G'}(U_2) \geq r_2$. This partition of V_0 satisfies the final condition of the lemma, completing the proof. \square

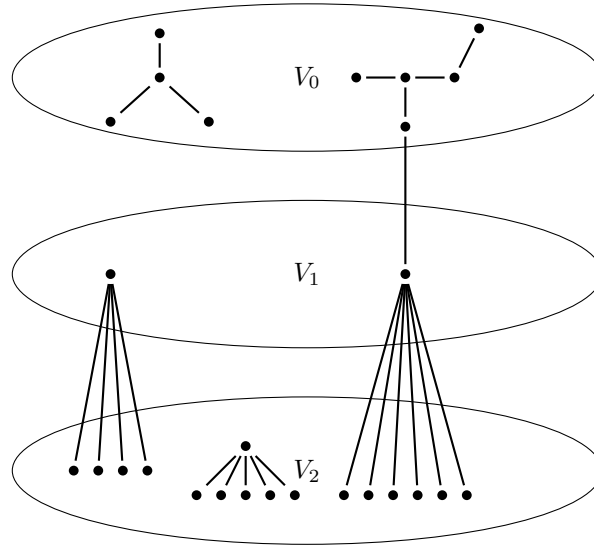


Figure 2: The structure of the forest F

5 Edge colouring a graph with large minimum degree

The aim of this section is to demonstrate an algorithm which, given a graph G with large minimum degree, uses the partition guaranteed by Lemma 14 to label G as Lemma 12 demands. In fact, we shall prove that Lemma 12 holds for k_1 and k_2 coprime odd integers, both at least 9, with the constant $c = 2k_1k_2 + k_2$.

For the rest of this section, we fix:

- coprime odd integers k_1 and k_2 , both at least 9,
- an integer δ ,
- a graph $G = (V, E)$ on n vertices with minimum degree at least δ ,

- $r_1 = (k_1 k_2 + 1)n$, $r_2 = (k_1 + 1)n$, and $r = \max(2(k_1 k_2 + k_1)n, m(n, r_1, r_1) + n)$,
- a choice of F_S , F , V_1 , U_1 and U_2 to satisfy the conclusions of Lemma 14 for δ , n , r , r_1 , r_2 and G .
- a label set L of size $|E|$, containing $[1, (2k_1 k_2 + k_2)n]$,
- and a function $g : V \mapsto \mathbb{N}$.

To prove Lemma 12 for k_1 and k_2 with the constant $c = 2k_1 k_2 + k_2$, our task is to show that G has a bijective g -antimagic colouring $f : E \mapsto L$ with no sum $s_{(f,g)}(v)$ divisible by $k_1 k_2$, provided δ is larger than some constant. Before we begin to label E , as before we define $V_0 = V \setminus V(F_S)$, and $V_2 = V(F_S) \setminus V_1$. Also, we define U_3 to be $\{v \in V_0 : E_G(\{v\}, V_0)\} \subseteq F\}$, and we write the stars of F_S as S_1, \dots, S_k , with centres c_1, \dots, c_k . Further, we define a new partition on the edge set of G as follows:

$$\begin{aligned} E_1 &= E_G(V_2, V) \setminus F, \\ E_2 &= E_G(V_1, V_0 \cup V_1) \setminus F, \\ E_3 &= E_G(V_0) \setminus F, \\ E_4 &= F \setminus E(F_S), \\ E_5 &= E(F_S). \end{aligned}$$

Note that the E_i are disjoint, and their union is E . Now we define a partition of the labels of L as follows:

$$\begin{aligned} L_1 &= [1, k_1 k_2 n] \setminus \{k_1 k_2 i : 1 \leq i \leq |F|\}, \\ L_2 &= [\max(L_1) + 1, \max(L_1) + k_1 k_2 (2|V_1| + |U_3|)], \\ L_3 &= [\max(L_2) + 1, \max(L_2) + (|V_0 \setminus U_3| - 1)(k_1 k_2 + k_2)], \\ L_4 &= \{k_1 k_2 i : 1 \leq i \leq |F|\}, \\ L_T &= L \setminus (L_1 \cup L_2 \cup L_3 \cup L_4). \end{aligned}$$

Note that the L_i are all disjoint. Also, note that at most half of the vertices in $V_1 \cup V_2$ are centres of stars in F_S , since every star has at least one vertex which is not the centre. Since V_1 is a subset of the centres of stars in F_S , we have $|V_1| \leq |V_2|$, and so

$$\begin{aligned} \max(L_1 \cup L_2 \cup L_3 \cup L_4) &= k_1 k_2 n + 2k_1 k_2 |V_1| + k_1 k_2 |U_3| + \\ &\quad + (k_1 k_2 + k_2)(|V_0 \setminus U_3| - 1) \\ &< k_1 k_2 n + k_1 k_2 |V_2 \cup V_1| + (k_1 k_2 + k_2) |V_0| \\ &\leq (2k_1 k_2 + k_2)n. \end{aligned}$$

Since $[1, cn] \subseteq L$, we have $L_1 \cup L_2 \cup L_3 \cup L_4 \subseteq L$. Also, by Condition 2 of Lemma 14 we have $|E_3| \geq (2k_1k_2 + k_2)n - |F| \geq |L_1 \cup L_2 \cup L_3|$. Hence we have

$$\begin{aligned} |L_T| &= |L| - |L_1 \cup L_2 \cup L_3| - |L_4| \\ &= |E| - |L_1 \cup L_2 \cup L_3| - |E_4| - |E_5| \\ &= |E_1 \cup E_2 \cup E_3| - |L_1 \cup L_2 \cup L_3| \\ &\geq |E_1 \cup E_2|. \end{aligned}$$

The L_i for $1 \leq i \leq 4$ are chosen carefully, and when we need some control over the label we use for an edge it will be these sets that are useful. The set L_T we have no control over whatsoever; we shall use labels from L_T when it is unimportant what label an edge receives.

Lemma	Edge set	Label set	Aim
15	E_1	Some of L_1 , some of L_T	Centres of stars in V_2 have sum equal to 1 (mod k_1k_2n), other vertices in V_2 have sum m (mod k_1k_2n)
16	E_2	Some of L_2 , some of L_T	Vertices in V_1 have sum equal to 1 (mod k_1k_2). Vertices in U_3 have sum not equal to 0 or 1 (mod k_1), and not too many in any class modulo k_1 .
17	E_3	All of L_3 , rest of L_1, L_2, L_T	Vertices in $U_1 \setminus U_3$ have sum not equal to 0 or 1 (mod k_1), and not too many in any class modulo k_1 , and similarly for $U_2 \setminus U_3$ and k_2 .
18	E_4	Some of L_4	Antimagic on V_0
20	E_5	Rest of L_4	Antimagic on centres of stars

Table 1: Strategy for labelling G

Now we are ready to begin labelling E . We shall do this by labelling the edge sets E_i in turn. We do this in a series of lemmas; each lemma takes the labelling guaranteed by the last, and extends it to label another set of edges. Table 1 summarises the edges labelled in each lemma, the labels used, and the aim for the partials sums of the labelling.

To give an overview of the labelling, we shall first go up the graph (as depicted in Figure 2), labelling all edges not in the spanning forest F . Here, we shall control the partial sums at vertices modulo k_1 and k_2 ; as we go up the graph, we shall be able to achieve less and less precise control over these sums. Then, we shall come back down the graph labelling F , essentially labelling greedily to avoid sums being equal. As we get nearer the end, we have fewer labels to choose from, but our more precise control over the partial sums so far and the special structure of F_S compensates for this lack of choice.

First we shall label the set E_1 , with labels from L_1 and L_T . Our aim here is that the vertices of V_2 receive specified partial sums modulo k_1k_2n ; the sums at

centres of stars will be congruent to 1, while the sums at other vertices will be congruent to 0 modulo k_1 and 1 modulo k_2 . We can achieve this without much difficulty, using two edges from a vertex $v \in V_2$ to $V_0 \cup V_1$ to fix the sum at v .

We define the integer m to be the unique element of $\{0, \dots, k_1 k_2 - 1\}$ with $m \equiv 0 \pmod{k_1}$ and $m \equiv 1 \pmod{k_2}$.

Lemma 15. *There is an injective labelling $f_1 : E_1 \mapsto L_1 \cup L_T$ such that*

1. *if v is a vertex of V_2 which is the centre c_i of one of the stars S_i , then $s_{(f_1, g)}(v) \equiv 1 \pmod{k_1 k_2 n}$,*
2. *if v is a vertex of V_2 which is not the centre of one of the stars S_i , then $s_{(f_1, g)}(v) \equiv m \pmod{k_1 k_2 n}$.*

Proof. By Condition 6 of Lemma 14, we can choose a set $E'_1 \subseteq E_1$ consisting of two edges from each $v \in V_2$ to $V_0 \cup V_1$. We label all edges in $E_1 \setminus E'_1$ with distinct and otherwise arbitrary labels from L_T . There are enough labels in L_T to do this, since $|L_T| > |E_1|$. Let $f'_1 : E_1 \setminus E'_1 \mapsto L_T$ be the labelling this gives us. Now we step through the vertices of V_2 in any order, labelling the edges of E'_1 adjacent to each vertex as we come to it. When we come to a vertex v , suppose the edges at v in E'_1 are e_1 and e_2 . We label e_1 and e_2 with labels l_1 and l_2 , obeying the following conditions:

1. l_1 and l_2 are not the same as any label already used in the labelling of E'_1 ,
2. l_1 and l_2 are labels of L_1 ,
3. $s_{(f'_1, g)}(v) + l_1 + l_2 \equiv \begin{cases} 1 \pmod{k_1 k_2 n} : & \text{if } v \text{ is } c_i \text{ for some } 1 \leq i \leq k \\ m \pmod{k_1 k_2 n} : & \text{otherwise.} \end{cases}$

We claim this is always possible. Indeed, when we reach a vertex $v \in V_2$, at most $2|V_2| - 2$ labels from L_1 have been used, so at least

$$k_1 k_2 n - |F| - 2|V_2| + 2 > (k_1 k_2 - 3)n > k_1 k_2 n / 2$$

labels remain, each of which is in a different congruency class modulo $k_1 k_2 n$. Hence there are two unused labels l_1 and l_2 in L_1 satisfying the conditions. Let $f'_1 : E'_1 \mapsto L_1$ be the labelling this process gives. We define f_1 to be f'_1 on E'_1 , and f'_1 on $E_1 \setminus E'_1$. Then f_1 is a labelling of E_1 satisfying the conditions of the lemma. \square

Next we take the labelling given by Lemma 15, and extend it to also label E_2 . E_2 will be labelled using labels from L_2 and L_T . Our aim here is that vertices in V_1 receive partial sums congruent to 1 modulo $k_1 k_2$, while the partial sums at vertices of U_3 are not congruent to 0 or 1 modulo k_1 , and there are not too many in any other congruency class modulo k_1 . This is achieved using Lemma 9, which precisely guarantees us a labelling of this form.

Lemma 16. *There is an injective labelling $f_2 : E_1 \cup E_2 \mapsto L_1 \cup L_2 \cup L_T$ such that*

1. if v is a vertex of V_2 which is the centre c_i of one of the stars S_i , then $s_{(f_2, g)}(v) \equiv 1 \pmod{k_1 k_2 n}$,
2. if v is a vertex of V_2 which is not the centre of one of the stars S_i , then $s_{(f_2, g)}(v) \equiv m \pmod{k_1 k_2 n}$,
3. if v is a vertex of V_1 , then $s_{(f_2, g)}(v) \equiv 1 \pmod{k_1 k_2}$,
4. the induced colouring of the vertices of U_3 satisfies the following conditions:
 - a) $n_{(f_2, g, U_3, k_1)}(0) = n_{(f_2, g, U_3, k_1)}(1) = 0$,
 - b) for each $2 \leq i \leq k_1 - 1$, $n_{(f_2, g, U_3, k_1)}(i) \leq |U_3|/(k_1 - 4) + 2k_1 - 3$.

Proof. First, applying Lemma 15 gives us an injective labelling $f_1 : E_1 \mapsto L_1 \cup L_T$ satisfying the conclusions of that lemma. Let L'_T be the set of labels in L_T which are not in the image of f_1 . Now, for a vertex $v \in V$, let $g'(v) = s_{(f_1, g)}(v)$. Define a graph G' with vertex set $V_0 \cup V_1$ and edge set E_2 . Now, we apply Lemma 9. In the statement of that lemma, we have a graph G , vertex sets A , B and B' , integers k_1 and k_2 , and a label set L ; here we use the graph G' , the sets V_1 , V_0 and U_3 , the integers k_1 and k_2 , and the label set $L' = L_2 \cup L'_T$. To check the conditions of Lemma 9 hold, by Condition 7 of Lemma 14 every vertex in V_1 has at least 2 edges to V_0 . Since vertices in U_3 have no edges to V_0 , by Condition 6 of Lemma 14 every vertex in U_3 has at least one edge to V_1 . Since $|L'_T| \geq |L_T| - |E_1| \geq |E_2|$, L' contains at least $|L_2| + |L'_T| \geq |E(G')| + 4k_1 k_2 |A| + k_1 |B'|$ labels. Finally, L_2 contains the required number of labels in each congruency class modulo k_1 and $k_1 k_2$, and so Lemma 9 does indeed apply. We set the function $g : V(G') \mapsto \mathbb{N}$ in that lemma to be g' , and we set the function $t : V(G') \mapsto \mathbb{N}$ to be constantly 1. Then by Lemma 9 there is an injective labelling $f'_2 : E_2 \mapsto L'$ such that

1. for each v in V_1 , $s_{(f'_2, g')}(v) \equiv 1 \pmod{k_1 k_2}$,
2. $n_{(f'_2, g', U_3, k_1)}(0) = n_{(f'_2, g', U_3, k_1)}(1) = 0$,
3. for each $2 \leq i \leq k_1 - 1$, $n_{(f'_2, g', U_3, k_1)}(i) \leq |U_3|/(k_1 - 4) + 2k_1 - 3$.

Now, we define the labelling $f_2 : E_1 \cup E_2 \mapsto L_1 \cup L_2 \cup L_T$ by setting $f_2 = f_1$ on E_1 and $f_2 = f'_2$ on E_2 . Since f'_2 does not label any edge incident with E_1 , the properties first two conditions of the lemma follow from the corresponding conditions for f_1 . Also, since for all vertices v we have $s_{(f_2, g)}(v) = s_{(f'_2, g')}(v)$, the other conditions in the lemma follow from the above conditions on f'_2 . \square

In the next lemma, we take the labelling given by Lemma 16, and extend it to also label E_3 . This will be done with the remainder of the sets L_1 , L_2 and L_T , and the whole of the label set L_3 . We define $U'_1 = U_1 \setminus U_3$ and $U'_2 = U_2 \setminus U_3$. The aim is that the partial sums at vertices in U'_1 are not congruent to 0 or 1 modulo k_1 , and there are not too many in any other congruency class modulo k_1 , and similarly for U'_2 and k_2 . To achieve this we shall use Lemma 8, which guarantees us a labelling to achieve precisely these conditions.

Lemma 17. *There is a bijective labelling $f_3 : E_1 \cup E_2 \cup E_3 \mapsto L_1 \cup L_2 \cup L_3 \cup L_T$ such that*

1. *if v is a vertex of V_2 which is the centre of a star in F_S , then $s_{(f_3,g)}(v) \equiv 1 \pmod{k_1 k_2 n}$,*
2. *if v is a vertex of V_2 which is the centre of a star in F_S , then $s_{(f_3,g)}(v) \equiv m \pmod{k_1 k_2 n}$,*
3. *if v is a vertex of V_1 , then $s_{(f_3,g)}(v) \equiv 1 \pmod{k_1 k_2}$,*
4. *the induced colouring of the vertices of U_3 satisfies the following conditions:*
 - a) $n_{(f_3,g,U_3,k_1)}(0) = n_{(f_3,g,U_3,k_1)}(1) = 0$,
 - b) *for each $2 \leq i \leq k_1 - 1$, $n_{(f_3,g,U_3,k_1)}(i) \leq |U_3|/(k_1 - 4) + 2k_1 - 3$,*
5. *for $i = 1$ and 2 , the induced colouring of the vertices of U'_i satisfies the following conditions:*
 - a) $n_{(f_3,g,U'_i,k_i)}(0) = n_{(f_3,g,U'_i,k_i)}(1) = 0$,
 - b) *for each $2 \leq j \leq k_i - 1$, $n_{(f_3,g,U'_i,k_i)}(j) \leq |U'_i|/(k_i - 3) + k_i + 2$.*

Proof. First, applying Lemma 16 gives us an injective labelling $f_2 : E_1 \cup E_2 \mapsto L_1 \cup L_2 \cup L_T$ satisfying the conclusions of that lemma. Let L' be those labels in $L_1 \cup L_2 \cup L_3 \cup L_T$ which are not in the image of f_2 . Note that since $|E_4| + |E_5| = |L_4|$, we have $|L'| = |E_3|$, and also note $L_3 \subseteq L'$. Let $g' : V \mapsto \mathbb{N}$ be given by $g'(v) = s_{(f_2,g)}(v)$. We define a graph G' , having vertex set $U'_1 \cup U'_2$, and edge set E_3 . We wish to label E_3 with L' using Lemma 8. In the statement of that lemma, we have a graph G , odd integers k_1 and k_2 , a label set L , and vertex sets V_1 and V_2 ; here we use the graph G' , the integers k_1 and k_2 , the label set L' and the vertex sets U'_1 and U'_2 . To show that Lemma 8 does indeed apply, note that since U'_1 and U'_2 are disjoint from U_3 , there are no isolated vertices in G' . Also, Condition 8 of Lemma 14 guarantees that $E_{G'}(U'_1) \geq (k_1 k_2 + 1)n \geq (k_1 k_2 + 1)|V(G')|$, and $|E_{G'}(U'_2)| \geq (k_1 + 1)n \geq (k_1 + 1)|V(G')|$. L' contains at least as many labels in each congruency class modulo $k_1 k_2$ and k_1 as Lemma 8 requires, since L_3 does and $L_3 \subseteq L'$. So Lemma 8 applies. We set the function $g : V \mapsto \mathbb{N}$ in that lemma to be g' . Then by Lemma 8 there is a bijective labelling $f'_3 : E_3 \mapsto L'$ such that for $i = 1$ and 2 we have:

1. $n_{(f'_3,g',U'_i,k_i)}(0) = n_{(f'_3,g',U'_i,k_i)}(1) = 0$,
2. *for each $2 \leq j \leq k_i - 1$, $n_{(f'_3,g',U'_i,k_i)}(j) \leq |U'_i|/(k_i - 3) + k_i + 2$.*

Now, we define the labelling $f_3 : E_1 \cup E_2 \cup E_3 \mapsto L_1 \cup L_2 \cup L_3 \cup L_T$ to be equal to f_2 on $E_1 \cup E_2$, and equal to f'_3 on E_3 . Since f'_3 labels no edge incident with V_1 , V_2 or U_3 , the properties we need for f_3 on those sets are inherited from the corresponding properties of f_2 . Also, for each $v \in U'_1 \cup U'_2$ we have $s_{(f'_3,g')}(v) = s_{(f_2,g)}(v)$. Hence the conditions we need on the sums in U'_1 and U'_2 follow from the above conditions on f'_3 . \square

At this stage, only the forest F remains unlabelled, and the labels $E_4 = \{k_1k_2, \dots, k_1k_2|F|\}$ remain to label F . In the next lemma, we extend the labelling from Lemma 17 to label E_4 as well. This shall be done using some of the labels from L_4 . The aim is that the vertices of V_0 receive distinct overall sums; to achieve this we shall use a greedy algorithm. This works because we have ensured that there are not too many vertices of V_0 with partial sums in any congruency class modulo k_1k_2 , and all the labels in E_4 are divisible by k_1k_2 , so each vertex in V_0 has a potential conflict with only fairly few other vertices in V_0 . It is at this stage that we shall need δ to be large, to guarantee E_5 is large and so that there are always enough labels remaining in L_4 to pick an appropriate one to label an edge.

Lemma 18. *Suppose the following equation holds for δ :*

$$\begin{aligned} \delta \left(1/2 - z(5, \delta) - \frac{2}{\min(k_1 - 4, k_2 - 3)} \right) &\geq \\ &\geq \max(2k_1k_2 + k_2, m'(k_1k_2 + 1, k_2 + 1) + 1) + 6k_1 + 2k_2 + 2. \end{aligned} \quad (3)$$

Then there is an injective labelling $f_4 : E_1 \cup E_2 \cup E_3 \cup E_4 \mapsto L$ such that:

1. *the image of f_4 includes $L_1 \cup L_2 \cup L_3 \cup L_T$,*
2. *if v is a vertex of V_2 which is the centre c_i of one of the stars S_i , then $s_{(f_4, g)} \equiv 1 \pmod{k_1k_2n}$,*
3. *if v is a vertex of V_2 which is not the centre of one of the stars S_i , then $s_{(f_4, g)} \equiv m \pmod{k_1k_2n}$,*
4. *if v is a vertex of V_1 , then $s_{(f_4, g)}(v) \equiv 1 \pmod{k_1k_2}$,*
5. *if v is a vertex of V_0 , $s_{(f_4, g)}(v)$ is not congruent to 0, 1, or m modulo k_1k_2 ,*
6. *for distinct vertices v_1 and $v_2 \in V_0$, $s_{(f_4, g)}(v_1) \neq s_{(f_4, g)}(v_2)$.*

Proof. First, applying Lemma 17 gives us a bijective labelling $f_3 : E_1 \cup E_2 \cup E_3 \mapsto L_1 \cup L_2 \cup L_3 \cup L_T$ satisfying the conclusions of that lemma. Now we label E_4 , using some of the labels from L_4 . We do this by stepping through the edges of E_4 in any order, labelling each as we reach it. Let $E_4 = \{e_1, \dots, e_r\}$. We define labellings f^i for $0 \leq i \leq r$, with f^i being a labelling $f^i : E_1 \cup E_2 \cup E_3 \cup \{e_1, \dots, e_i\} \mapsto L$. For $i = 0$, we define f^0 to be equal to f_3 . For $i \geq 1$, define f^i by setting $f^i = f^{i-1}$ on $E_1 \cup E_2 \cup E_3 \cup \{e_1, \dots, e_{i-1}\}$, and letting $f^i(e_i) = l$ for any label l satisfying the following conditions:

1. l is in L_4 , and not in the image of f^{i-1} ,
2. if $v, v' \in V_0$ with $v \in e$ and $v' \notin e$, $s_{(f^{i-1}, g)}(v) + l \neq s_{(f^{i-1}, g)}(v')$.

We claim that such a label always exists. When we reach the edge $e = v_1v_2$, there are at least $|E_5| + 1$ edges unlabelled, and correspondingly there are at least $|E_5| + 1$ labels which obey the first condition. Of these, Condition 2 rules out at most one label for each $v, v' \in V_0$ with $v \in e, v' \notin e$ and $s_{(f^{i-1},g)}(v) - s_{(f^{i-1},g)}(v') \in L_4$. An upper bound for the number of such vertices is the number of vertices v' in $V_0 \setminus e$ with $s_{(f_3,g)}(v')$ equal to $s_{(f_3,g)}(v_1)$ or $s_{(f_3,g)}(v_2)$ modulo k_1k_2 , since all labels in L_4 are divisible by k_1k_2 . From the conclusion of Lemma 17, the number of such vertices v' is at most

$$2 \left(\frac{|U_3|}{k_1 - 4} + 2k_1 - 3 + \frac{|U'_1|}{k_1 - 3} + k_1 + 2 + \frac{|U'_2|}{k_2 - 3} + k_2 + 2 \right) - 2.$$

Indeed, the first term in the bracket represents the largest possible number of vertices $v \in U_3$ with $s_{(f_3,g)}(v) \equiv s_{(f_3,g)}(v_1) \pmod{k_1}$, the second the largest possible number of vertices $v \in U'_1$ with $s_{(f_3,g)}(v) \equiv s_{(f_3,g)}(v_1) \pmod{k_1}$, and the third the largest possible number of vertices $v \in U'_2$ with $s_{(f_3,g)}(v) \equiv s_{(f_3,g)}(v_1) \pmod{k_2}$. We may subtract 2 because we need not consider the vertices v_1 and v_2 . So there is a label that obeys the conditions so long as

$$|E_5| + 1 \geq 2 \left(\frac{|U_3|}{k_1 - 4} + \frac{|U'_1|}{k_1 - 3} + \frac{|U'_2|}{k_2 - 3} \right) + 6k_1 + 2k_2. \quad (4)$$

Assume for now that this equation holds. We define the labelling on $f_4 : E_1 \cup E_2 \cup E_3 \cup E_4 \mapsto L$ to be equal to f^r . We claim that f_4 satisfies the conditions of the lemma.

The first condition is satisfied, since the image of f_4 includes the image of f_3 , which is $L_1 \cup L_2 \cup L_3 \cup L_T$. The second and third conditions are guaranteed by the equivalent conditions for f_3 , since E_4 has no edge incident with V_1 . The fourth and fifth conditions hold for f_3 , and so also for f_4 , as E_4 is entirely labelled with labels divisible by k_1k_2 . For the final condition, we claim the restrictions on the labelling of E_5 guarantee that $s_{(f_4,g)}(v_1) \neq s_{(f_4,g)}(v_2)$ for distinct vertices v_1 and v_2 in V_0 . Indeed, let e_j be the last edge incident with precisely one of v_1 and v_2 to be labelled; such an edge exists, by Condition 4 of Lemma 14. The conditions on the label given to e_j guarantee that $s_{(f^j,g)}(v_1) \neq s_{(f^j,g)}(v_2)$, and hence $s_{(f_4,g)}(v_1) \neq s_{(f_4,g)}(v_2)$.

To prove the lemma, it remains to check (4). From Condition 3 of Lemma 14, we have

$$|E_5| = \left| \bigcup_{i=1}^k E(S_i) \right| \geq n(1/2 - z(5, \delta) - 2/\delta) - 1 - r/\delta. \quad (5)$$

So to check (4) holds, it suffices to show that

$$n(1/2 - z(5, \delta) - 2/\delta) - r/\delta \geq 2 \left(\frac{|U_3|}{k_1 - 4} + \frac{|U'_1|}{k_1 - 3} + \frac{|U'_2|}{k_2 - 3} \right) + 6k_1 + 2k_2. \quad (6)$$

Now, $|U_3| + |U'_1| + |U'_2| = |V_0| \leq n$, and so to establish (6) it suffices to show that

$$n \left(1/2 - z(5, \delta) - \frac{2}{\min(k_1 - 4, k_2 - 3)} - 2/\delta \right) - r/\delta \geq 6k_1 + 2k_2. \quad (7)$$

Rearranging this equation, and multiplying by δ/n , (7) is equivalent to

$$\delta/2 - \delta z(5, \delta) - \frac{2\delta}{\min(k_1 - 4, k_2 - 3)} \geq r/n + (6k_1 + 2k_2)\delta/n + 2. \quad (8)$$

However, G is a graph on n vertices with minimum degree at least δ , so $\delta/n < 1$, and so for (8) to hold it is enough that

$$\delta \left(1/2 - z(5, \delta) - \frac{2}{\min(k_1 - 4, k_2 - 3)} \right) \geq r/n + 6k_1 + 2k_2 + 2. \quad (9)$$

Now, $r = \max((2k_1k_2 + k_2)n, m(n, r_1, r_2) + n)$; but from Corollary 4 we have $m(n, r_1, r_2) = m(n, (k_1k_2 + 1)n, (k_1 + 1)n) \leq m'(k_1k_2 + 1, k_1 + 1)n$. So for (9) to hold it is enough that

$$\begin{aligned} & \delta \left(1/2 - z(5, \delta) - \frac{2}{\min(k_1 - 4, k_2 - 3)} \right) \geq \\ & \geq \max(2k_1k_2 + k_2, m'(k_1k_2 + 1, k_2 + 1) + 1) + 6k_1 + 2k_2 + 2. \end{aligned}$$

This is precisely the assumption of the lemma, and the proof is complete. \square

Lemma 18 leaves only E_5 unlabelled, and the unused labels are a subset of L_4 . We wish label E_5 so that the sums at the centres of the stars in F_S are all distinct. To achieve this, we use the following simple lemma:

Lemma 19. *Let S_1, \dots, S_k be vertex disjoint stars with centres c_1, \dots, c_k , let L be a set of integers of size $|\bigcup_{i=1}^k E(S_i)|$, and let g be a function $g : \bigcup_{i=1}^k V(S_i) \mapsto \mathbb{N}$. Then there exists a bijective edge labelling $f : \bigcup_{i=1}^k E(S_i) \mapsto L$ such that the sums $s_{(f,g)}(c_i)$ are distinct.*

Proof. We prove this by induction on k ; for $k = 1$ the result is trivial. For $k \geq 2$, let $L = \{l_1, \dots, l_r\}$ with $l_1 < \dots < l_r$. For $1 \leq i \leq k$ let $n_i = g(c_i) + \sum_{j=1}^{|E(S_i)|} l_j$. Without loss of generality, n_k is the smallest of the n_i . We label $E(S_k)$ with $\{l_1, \dots, l_{|E(S_k)|}\}$ in any order. By the induction hypothesis, there is a labelling of $\bigcup_{i=1}^{k-1} E(S_i)$ with the rest of L so that the sums at c_1, \dots, c_{k-1} are distinct. Also, for this labelling we have $s_{(f,g)}(c_i) > n_i \geq n_k = s_{(f,g)}(c_k)$ for $i \neq k$, and so in fact the sums at the centres of all the stars are distinct. \square

Using the labelling guaranteed by Lemma 18 and applying Lemma 19 to label E_5 , the edge set of the stars in F_S , we show that we can find an antimagic colouring of G :

Lemma 20. *Suppose δ satisfies (3). Then there is a bijective labelling $f_5 : E \mapsto L$ so that f_5 is g -antimagic, and for all $v \in V$ we have $s_{(f_5,g)}(v) \not\equiv 0 \pmod{k_1k_2}$.*

Proof. First we apply Lemma 18 to G — let $f_4 : E_1 \cup E_2 \cup E_3 \cup E_4 \mapsto L$ be a labelling satisfying the conclusions of that lemma, and let L' be the labels not in the image of f_4 ; so $L' \subseteq L_4$. We have $|L'| = |E_5|$. For a vertex $v \in V$, let $g'(v) = s_{(f_4, g)}(v)$. Now, we apply Lemma 19 to the stars S_1, \dots, S_k which make up F_S , and the label set L' , with the function $g' : \bigcup_{i=1}^k V(S_i) \mapsto \mathbb{N}$ for the function g . This gives us a bijective labelling $f'_5 : E_5 \mapsto L'$ so that $s_{(f'_5, g')}(c_i) \neq s_{(f'_5, g')}(c_j)$ for $1 \leq i < j \leq k$. Now, let f_5 be the labelling given by f_4 on $E_1 \cup E_2 \cup E_3 \cup E_4$ and f'_5 on E_5 . We claim that f_5 is a g -antimagic labelling, with no sum $s_{(f_5, g)}(v)$ divisible by $k_1 k_2$. Since f'_5 labels no edge incident with V_0 , and all the labels used by f'_5 are divisible by $k_1 k_2$, the following conditions hold from the properties of f_4 :

1. if v is a vertex of V_2 which is the centre c_i of one of the stars S_i , then $s_{(f_5, g)} \equiv 1 \pmod{k_1 k_2}$,
2. if v is a vertex of V_2 which is not the centre of one of the stars S_i , then $s_{(f_5, g)} \equiv m \pmod{k_1 k_2}$,
3. if v is a vertex of V_1 , then $s_{(f_5, g)} \equiv 1 \pmod{k_1 k_2}$,
4. if v is a vertex of V_0 , $s_{(f_5, g)}$ is not congruent to 0, 1 or m modulo $k_1 k_2$,
5. for distinct vertices v_1 and $v_2 \in V_0$, $s_{(f_5, g)}(v_1) \neq s_{(f_5, g)}(v_2)$.

Since by Condition 1 of Lemma 14 all the vertices of V_1 are the centres of stars, all centres c of stars in F_S have $s_{(f_5, g)}(c) \equiv 1 \pmod{k_1 k_2}$, and all the other vertices v in V_2 have $s_{(f_5, g)}(v) \equiv m \pmod{k_1 k_2}$, whereas all vertices v in V_0 have $s_{(f_5, g)}(v) \notin \{0, 1, m\} \pmod{k_1 k_2}$. So it suffices to check that none of these three sets contain two vertices with equal sums. For two vertices in V_0 , this is the last condition above. For two vertices c_i and c_j which are the centres of stars S_i and S_j , from our application of Lemma 19 we have $s_{(f_5, g)}(c_i) = s_{(f'_5, g')}(c_i) \neq s_{(f'_5, g')}(c_j) = s_{(f_5, g)}(c_j)$. For two vertices v_1 and v_2 in V_2 which are not the centres of stars, each has exactly one edge in a star, and so we have $s_{(f_5, g)}(v_1) = s_{(f_4, g)}(v_1) + l_1$, and $s_{(f_5, g)}(v_2) = s_{(f_4, g)}(v_2) + l_2$, for some $l_1 \neq l_2$ in L_4 , and hence in $[1, k_1 k_2 n]$. However, from Lemma 18 we also have $s_{(f_4, g)}(v_1) \equiv s_{(f_4, g)}(v_2) \equiv m \pmod{k_1 k_2 n}$. Hence $s_{(f_5, g)}(v_1) \neq s_{(f_5, g)}(v_2)$. \square

From the bound on $z(k, l)$ given by Theorem 5, it is easy to see that (3) holds for sufficiently large δ so long as $\frac{2}{\min(k_1 - 4, k_2 - 3)} < 1/2$. This establishes Lemma 12 for $k_1, k_2 \geq 9$, in which case we can take c to be $2k_1 k_2 + k_2$ and δ to be the minimal integer satisfying (3). In fact, it can be calculated that the best bound on d is achieved when $(k_1, k_2) = (13, 11)$, for which Lemma 12 holds with constants $c = 2k_1 k_2 + k_2 = 297$ and $\delta = 1663$. Then from our proof of Theorem 1 from Lemma 12, Theorem 1 holds with $d_0 = 4182$.

6 Further Work

In this section, we discuss possible directions for work on antimagic graphs. The main open question remains the conjecture of Hartsfield and Ringel, that all connected graphs other than K_2 are antimagic. However, Theorem 1 does not require G to be connected, leading us to pose the following question:

Question 1. *What is the least real number d_0 such that any graph with average degree at least d_0 with no isolated edges and at most one isolated vertex is antimagic?*

Our proof gives an upper bound on d_0 of 4182. While there may be room for decreasing this bound by proceeding more carefully, it seems unlikely that an approach similar to the one employed here will bring the bound below, say, 1000. For a lower bound, it is easy to see that if G has no isolated vertices and

$$|E|(|E| + 1) < |V|(|V| + 1)/2, \quad (10)$$

then G is not antimagic — indeed, the total of the induced sums at all the vertices is not large enough for the vertex sums to be distinct positive integers. This gives $d_0 \geq \sqrt{2}$. We conjecture that in fact any graph with no isolated edges and at most one isolated vertex not satisfying (10) is antimagic, and hence that $d_0 = \sqrt{2}$.

Another direction arises from the observation that our proof of Theorem 1 allows us to construct antimagic labellings of graphs G with large average degree with many more label sets than just $[1, |E(G)|]$. In fact, we approximately need one label in each congruency class modulo $k_1 k_2 n$, and a further n in each congruency class modulo $k_1 k_2$. This leads us to ask whether we could use any label set. Explicitly, we call a graph $G = (V, E)$ *label-antimagic* if for any set L of positive integers with $|L| = |E|$ there is a bijective antimagic labelling $f : E \mapsto L$.

Question 2. *Is there some constant d_l such that all graphs with average degree at least d_l with no isolated edges and at most one isolated vertex are label-antimagic?*

7 Acknowledgements

I would like to thank Karen Gunderson, who introduced me to this problem and with whom I had several helpful conversations about antimagic colourings. I would also like to thank Béla Bollobás and both the anonymous referees, whose comments have made the proof clearer and stronger.

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